Not all types of temperature changes in a solid continuum result in creation of thermal stresses. The chapter begins with the discussion of the condition on what type of temperature distribution causes thermal stresses. The analogy of temperature gradient with body forces is stated. Then the theoretical analysis of thermal stress problems is presented in three main classical coordinate systems, that is, the rectangular Cartesian coordinates, the cylindrical coordinates, and the spherical coordinates. In discussing the analytical methods of solution, Airy stress function, Boussinesq function, the displacement potential, Michell function, and Papkovich functions are defined and the general solution in three coordinate systems are given in terms of these functions.

1 Introduction

In this chapter, some basic problems of the theory of thermoelasticity are discussed. The answers will be given to the following important questions:

1. Do all kinds of distribution of the temperature in a solid body create thermal stresses?

2. What are the relations between the temperature distribution, the boundary conditions, and the thermal stresses?

3. Can we relate the thermal stresses to the body forces?

In addition, will be presented some general methods of solutions of problems of thermoelasticity in two and three dimensions. Navier equations will be derived
in three coordinate systems, namely, in the rectangular Cartesian coordinates, the cylindrical coordinates, and the spherical coordinates. The general solution of the set of governing equations in each of these coordinate systems will be given. Although the final solutions depend on specific boundary conditions, the general solutions in this chapter will be given without the consideration of the effect of the boundary conditions. However, in the following chapters, some proposed methods of solution will be applied to physical problems with boundary conditions stated.

2 Temperature Distribution for Zero Thermal Stress

Generally, when a body is exposed to a thermal gradient, it is expected that thermal stresses will be developed within the body. The question arises whether any type of thermal gradient results in thermal stresses.

Consider a freely supported body, so that no constraint prevents its thermal expansion. We further assume that the boundary traction \( \bar{t}_i^* \) and the body force \( X_i \) are zero. Setting all the stress components equal to zero, \( \sigma_{ij} = 0 \), the surface boundary condition

\[
\bar{t}_i^* = n_j \sigma_{ij} \quad (3.2-1)
\]

is identically satisfied, and the governing equation in terms of the stresses, Eq. (1.11-18), reduces to

\[
T_{,ij} + \frac{3\lambda + 2\mu}{\lambda + 2\mu} T_{,kk}\delta_{ij} = 0 \quad (3.2-2)
\]

In the expanded form, this equation reads

\[
\begin{align*}
(1+\nu)\nabla^2 T + \frac{\partial^2 T}{\partial x^2} &= 0 \\
(1+\nu)\nabla^2 T + \frac{\partial^2 T}{\partial y^2} &= 0 \\
(1+\nu)\nabla^2 T + \frac{\partial^2 T}{\partial z^2} &= 0 \\
\frac{\partial^2 T}{\partial x\partial y} &= 0, \quad \frac{\partial^2 T}{\partial y\partial z} = 0, \quad \frac{\partial^2 T}{\partial z\partial x} = 0 \quad (3.2-3)
\end{align*}
\]

Adding up the first three equations results in \( \nabla^2 T = 0 \). This means that the only possible temperature distribution which produces zero thermal stresses in a body of simply connected region is when

\[
\begin{align*}
\nabla^2 T &= 0 \\
\frac{\partial^2 T}{\partial x\partial y} &= \frac{\partial^2 T}{\partial y\partial z} = \frac{\partial^2 T}{\partial z\partial x} = 0 \quad (3.2-4)
\end{align*}
\]
The unique solution for the temperature distribution satisfying Eq. (3.2-4) is

\[ T - T_0 = a + bx + cy + dz = B_0 + B_i x_i \]  

(3.2-5)

where \( a, b, c, \) and \( d \) are some arbitrary constants of integration and \( B_0 = a, B_1 = b, B_2 = c, \) and \( B_3 = d. \) A temperature distribution of this form will not produce any thermal stresses in a body of simply connected region provided that the body has not been constrained by its boundary in any direction.

For multiply connected region, in addition to Eq. (1.11-18), Cesàro integral equations (1.11-21) and (1.11-22) must be satisfied for zero thermal stresses. Equation (1.11-21) for \( \sigma_{ij} = 0 \) reduces to \[ \oint_{C_s} [(T - T_0) \delta_{il} - e_{jik} e_{kml} T_m] dx_l = 0 \]

(3.2-6)

Substituting for temperature from Eq. (3.2-5) and using the identity on permutation symbol, Eq. (1.8-13), this equation yields

\[ B_0 \oint_{C_s} dx_l + B_i \oint_{C_s} B_i x_i dx_l - B_k \oint_{C_s} B_k x_k dx_l + B_l \oint_{C_s} B_l x_m dx_m = 0 \]  

(3.2-7)

This equation is identically satisfied as the second and third terms cancel and the first and last integrals on the closed curve \( C_s \) are zero.

The second Cesàro integral equation for \( \sigma_{ij} = 0, \) from Eq. (1.11-22), yields

\[ \oint_{C_s} e_{kml} T_m dx_l = 0 \] 

(3.2-8)

Substituting \( T \) from Eq. (3.2-5) gives

\[ \oint_{C_s} e_{kml} B_m dx_l = 0 \] 

(3.2-9)

This equation is clearly satisfied. It is, therefore, concluded that a linear distribution of temperature in a body, of either simply or multiply connected region, results in zero thermal stresses provided that the boundaries are free of traction.

The displacement components, not including the rigid body motion, corresponding to the temperature distribution (3.2-5) are obtained from the strain-displacement relations. Since the stresses are zero, then

\[ \epsilon_{xx} = \frac{\partial u}{\partial x} = \alpha (T - T_0) = \alpha (a + bx + cy + dz) \]

\[ \epsilon_{yy} = \frac{\partial v}{\partial y} = \alpha (T - T_0) = \alpha (a + bx + cy + dz) \]

\[ \epsilon_{zz} = \frac{\partial w}{\partial z} = \alpha (T - T_0) = \alpha (a + bx + cy + dz) \]

\[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 \] 

(3.2-10)
Integration of these partial differential equations, following a procedure described by Love [2] (page 127), and by Timoshenko and Goodier [3] (page 252), yields

\[
\begin{align*}
  u &= \alpha [(a + bx + cy + dz)x - \frac{b}{2}(x^2 + y^2 + z^2)] \\
  v &= \alpha [(a + bx + cy + dz)y - \frac{c}{2}(x^2 + y^2 + z^2)] \\
  w &= \alpha [(a + bx + cy + dz)z - \frac{d}{2}(x^2 + y^2 + z^2)] 
\end{align*}
\] (3.2-11)

The condition on temperature distribution for a two-dimensional plane stress or plane strain problem to have zero thermal stresses is obtained by setting the stress components equal to zero in the proper governing equations. We assume that in either case, the temperature distribution is independent of \( z \), and is a function of \( x \) and \( y \). Furthermore, we will first establish the condition for simply connected regions.

**Plane stress problems**

For plane stress problems the condition is obtained by setting in Eq. (1.12-9)

\[
\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0
\] (3.2-12)

Since the body force and the inertia terms are also excluded, Eq. (1.12-9) for zero thermal stress condition reduces to

\[
\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0
\] (3.2-13)

In this case the strains are

\[
\begin{align*}
  \epsilon_{xx} &= \epsilon_{yy} = \epsilon_{zz} = \alpha(T - T_0) \\
  \epsilon_{xy} &= \epsilon_{yz} = \epsilon_{zx} = 0
\end{align*}
\] (3.2-14)

**Plane strain problems**

For simple plane strain problems, from Eq. (1.12-26), the condition for zero in-plane stresses, \( \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0 \), is

\[
\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0
\] (3.2-15)

The stress in \( z \)-direction is

\[
\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) - E\alpha(T - T_0) = -E\alpha(T - T_0)
\] (3.2-16)
That is, the harmonic temperature distribution satisfying Eq. (3.2-15) results in zero in-plane stresses, but non-zero $\sigma_{zz}$. The strains for this case, from Eq. (1.12-24), are

$$
\begin{align*}
\epsilon_{zz} &= 0 \\
\epsilon_{xx} &= \epsilon_{yy} = (1 + \nu)\alpha(T - T_0) \\
\epsilon_{xy} &= \epsilon_{yz} = \epsilon_{zx} = 0
\end{align*}
$$

(3.2-17)

For multiply connected regions, the temperature distribution causing zero thermal stresses should satisfy Michell conditions in addition to the harmonic equation (3.2-15). Setting the stress components equal to zero, Eq. (1.13-12) for plane the stress condition gives

$$
\oint_{C_s} \frac{\partial T}{\partial n} ds = 0
$$

(3.2-18)

In addition, Eqs. (1.13-18) and (1.13-20) yield

$$
\oint_{C_s} (x \frac{\partial T}{\partial s} - y \frac{\partial T}{\partial n}) ds = \oint_{C_s} (y \frac{\partial T}{\partial s} + x \frac{\partial T}{\partial n}) ds = 0
$$

(3.2-19)

Equation (3.2-18) means that the heat flux through each of the interior holes must be zero. The same conditions are obtained for the plane strain condition. Therefore, the temperature distribution causing zero thermal stresses in the plane stress and plane strain conditions of a multiply connected region must satisfy Eqs. (3.2-15), (3.2-18), and (3.2-19).

The temperature distribution for the three-dimensional problems in which the thermal displacements are zero is easily obtained by considering

$$
\begin{align*}
u = v = w = 0
\end{align*}
$$

(3.2-20)

This yields

$$
\begin{align*}
\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = \epsilon_{xy} = \epsilon_{yz} = \epsilon_{zx} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0
\end{align*}
$$

(3.2-21)

From the stress-strain relations, Eq. (1.9-5), it follows that the corresponding stresses are

$$
\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -\frac{E\alpha(T - T_0)}{1 - 2\nu}
$$

(3.2-22)

This result is equivalent to a state of hydrostatic stresses for an incompressible material where the normal components of the stresses in any direction are equal. This situation occurs when a body is heated and is prevented to expand in any direction.
3 Analogy of Thermal Gradient with Body Forces

The problems of thermal stresses can be formulated in such a way that the effects of thermal gradient are considered as the body forces. This method reduces the problem to those of elasticity problems in the presence of body forces. For this purpose, we may consider a pure thermal stress problem, where body and surface forces are assumed to be zero. The equations of motion (1.10-3) for the static condition and in the absence of body forces are

\[
\begin{align*}
(\lambda + \mu) \frac{\partial e}{\partial x} + \mu \nabla^2 u - \frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial x} &= 0 \\
(\lambda + \mu) \frac{\partial e}{\partial y} + \mu \nabla^2 v - \frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial y} &= 0 \\
(\lambda + \mu) \frac{\partial e}{\partial z} + \mu \nabla^2 w - \frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial z} &= 0
\end{align*}
\]

(3.3-1)

where \( e = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \). The boundary conditions, from Eq. (1.10-6) with \( r_i^n = 0 \) are

\[
\begin{align*}
\lambda e_n x + G(n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y} + n_z \frac{\partial u}{\partial z}) + G(n_x \frac{\partial v}{\partial y} + n_y \frac{\partial v}{\partial y} + n_z \frac{\partial v}{\partial z}) + G(n_x \frac{\partial w}{\partial z} + n_y \frac{\partial w}{\partial z} + n_z \frac{\partial w}{\partial z}) \\
- \frac{E\alpha(T - T_0)}{1 - 2\nu} n_x &= 0 \\
\lambda e_n y + G(n_x \frac{\partial v}{\partial x} + n_y \frac{\partial v}{\partial y} + n_z \frac{\partial v}{\partial z}) + G(n_x \frac{\partial u}{\partial y} + n_y \frac{\partial u}{\partial y} + n_z \frac{\partial u}{\partial y}) + G(n_x \frac{\partial w}{\partial z} + n_y \frac{\partial w}{\partial z} + n_z \frac{\partial w}{\partial z}) \\
- \frac{E\alpha(T - T_0)}{1 - 2\nu} n_y &= 0 \\
\lambda e_n z + G(n_x \frac{\partial w}{\partial x} + n_y \frac{\partial w}{\partial y} + n_z \frac{\partial w}{\partial z}) + G(n_x \frac{\partial u}{\partial z} + n_y \frac{\partial u}{\partial z} + n_z \frac{\partial u}{\partial z}) + G(n_x \frac{\partial v}{\partial z} + n_y \frac{\partial v}{\partial z} + n_z \frac{\partial v}{\partial z}) \\
- \frac{E\alpha(T - T_0)}{1 - 2\nu} n_z &= 0
\end{align*}
\]

(3.3-2)

where \( n_x, n_y, \) and \( n_z \) are the cosine directors of the unit outer normal vector to the boundary. Comparison of these equations with those of elasticity equations reveals that a thermal stress problem may be treated as an elasticity (non-thermal) problem in the presence of body forces if an equivalent body force with components

\[
\begin{align*}
\frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial x} \\
\frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial y} \\
\frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial z}
\end{align*}
\]

in \( x, y, \) and \( z \)-directions, respectively, is considered. It is similarly concluded that the term \( \frac{E\alpha(T - T_0)}{1 - 2\nu} \) in Eqs. (3.3-2) is equivalent to the surface
force acting on the boundary in an elasticity (non-thermal) problem. In other
words, thermal stress problems are similar to elasticity problems where a body
is in the state of uniform constant temperature, but placed in a body force
field equivalent to a hydrostatic state of stress

\[ p = -\frac{E\alpha}{1 - 2\nu}(T - T_0) \]  

(3.3-3)

The stresses obtained by this equivalent body force should be superimposed
on the stresses obtained by stress-strain relations without thermal expansion.

Equation (3.3-3) has another significant property in interpreting the ther-
mal effects in solid bodies. Consider a small cube of an elastic solid under a
uniform temperature change \((T - T_0)\). If we allow the cube to expand freely,
the thermal strains are

\[
\begin{align*}
\epsilon_{xx} &= \epsilon_{yy} = \epsilon_{zz} = \alpha(T - T_0) \\
\epsilon_{xy} &= \epsilon_{yz} = \epsilon_{zx} = 0
\end{align*}
\]

In order to bring back the sides of the cube to its original length, a hydrostatic
pressure must act on all sides of the cube. It is easily verified that the pressure
given in Eq. (3.3-3) is the hydrostatic pressure which forces the cube to its
original size and, therefore, brings all the thermal expansions to zero. It should
be noticed, however, that the dimensions of the hydrostatic pressure are not
the same as that of body force, the former one being force per unit area, and
the latter being force per unit volume.

Let us now consider a heated body with zero body forces and free of sur-
face traction, but subjected to some arbitrary thermal gradient. The thermal
stresses can be calculated from Eqs. (3.3-1) once the displacement compo-
ents are obtained, using the boundary condition (3.3-2). This problem may
be treated as an isothermal elasticity problem provided we make a transfor-
mation of the stresses

\[
\begin{align*}
\sigma_{xx} &= \sigma'_{xx} - \frac{E\alpha(T - T_0)}{1 - 2\nu} \\
\sigma_{yy} &= \sigma'_{yy} - \frac{E\alpha(T - T_0)}{1 - 2\nu} \\
\sigma_{zz} &= \sigma'_{zz} - \frac{E\alpha(T - T_0)}{1 - 2\nu} \\
\sigma_{xy} &= \sigma'_{xy} \\
\sigma_{yz} &= \sigma'_{yz} \\
\sigma_{zx} &= \sigma'_{zx}
\end{align*}
\]  

(3.3-4)

where non-primed stresses are the real existing stresses in a heated body, and
primed stresses are the stresses in an unheated body which is subjected to the
equivalent body force. The temperature distribution in Eqs. (3.3-4) may be
as general as \(T = T(x, y, z)\). The relations between the strains and stresses
resulting from the equivalent body forces are

\[
\begin{align*}
\epsilon_{xx} &= \frac{1}{E} \left[ \sigma'_{xx} - \nu (\sigma'_{yy} + \sigma'_{zz}) \right] \\
\epsilon_{yy} &= \frac{1}{E} \left[ \sigma'_{yy} - \nu (\sigma'_{xx} + \sigma'_{zz}) \right] \\
\epsilon_{zz} &= \frac{1}{E} \left[ \sigma'_{zz} - \nu (\sigma'_{xx} + \sigma'_{yy}) \right] \\
\epsilon_{xy} &= \frac{1}{2G} \sigma'_{xy} \\
\epsilon_{yz} &= \frac{1}{2G} \sigma'_{yz} \\
\epsilon_{zx} &= \frac{1}{2G} \sigma'_{zx}
\end{align*}
\] (3.3-5)

As expected, relations between \( \epsilon_{ij} \) and \( \sigma'_{ij} \) exclude the term \( \alpha(T - T_0) \) which exists in thermal stress problems, because the effect of temperature is now included in the body force. Substitution of Eqs. (3.3-4) in the equation of motion (1.3-8), with \( X = Y = Z = 0 \) and \( \ddot{\rho}u_i = 0 \) gives

\[
\begin{align*}
\frac{\partial \sigma'_{xx}}{\partial x} + \frac{\partial \sigma'_{xy}}{\partial y} + \frac{\partial \sigma'_{xz}}{\partial z} - \frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial x} &= 0 \\
\frac{\partial \sigma'_{xy}}{\partial x} + \frac{\partial \sigma'_{yy}}{\partial y} + \frac{\partial \sigma'_{yz}}{\partial z} - \frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial y} &= 0 \\
\frac{\partial \sigma'_{xz}}{\partial x} + \frac{\partial \sigma'_{yz}}{\partial y} + \frac{\partial \sigma'_{zz}}{\partial z} - \frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial z} &= 0
\end{align*}
\] (3.3-6)

The boundary conditions in terms of new stresses become

\[
\begin{align*}
\sigma'_{xx} n_x + \sigma'_{xy} n_y + \sigma'_{xz} n_z &= \frac{E\alpha(T - T_0)}{1 - 2\nu} n_x \\
\sigma'_{yx} n_x + \sigma'_{yy} n_y + \sigma'_{yz} n_z &= \frac{E\alpha(T - T_0)}{1 - 2\nu} n_y \\
\sigma'_{zx} n_x + \sigma'_{zy} n_y + \sigma'_{zz} n_z &= \frac{E\alpha(T - T_0)}{1 - 2\nu} n_z
\end{align*}
\] (3.3-7)

A review of Eqs. (3.3-5) to (3.3-7) shows that a thermal stress problem can be solved as an isothermal elasticity problem provided that we take a body force field with components in \( x, y \), and \( z \)-directions given by

\[
\begin{align*}
X' &= \frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial x} \\
Y' &= \frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial y} \\
Z' &= \frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial z}
\end{align*}
\] (3.3-8)

and a surface traction with components

\[
\bar{X}' = \frac{E\alpha(T - T_0)}{1 - 2\nu} n_x
\]
4. General Solution of Thermoelastic Problems

Equilibrium equations in terms of displacement components were obtained to be, see Eq. (3.3-1),

\[
(\lambda + \mu) \frac{\partial e}{\partial x} + \mu \nabla^2 u = \frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial x},
\]

\[
(\lambda + \mu) \frac{\partial e}{\partial y} + \mu \nabla^2 v = \frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial y},
\]

\[
(\lambda + \mu) \frac{\partial e}{\partial z} + \mu \nabla^2 w = \frac{E\alpha}{1 - 2\nu} \frac{\partial T}{\partial z},
\]

(3.4-1)

where the temperature distribution \( T \) is assumed to be known from heat transfer equations. The complete solution of these equations may be represented as

\[
u_i = u_i^g + u_i^T \quad (i = 1, 2, 3)
\]

(3.4-2)

where \( u_i^g \) is the general solution of Eqs. (3.4-1) with the right-hand terms equal zero, and \( u_i^T \) is a particular solution. The general solution of Eqs. (3.4-1) with the right-hand terms equal zero was proposed by Papkovich [4,5] in the form

\[
u_i^g = \frac{\partial}{\partial x_i} (x_k \Phi_k) - 4(1 - \nu)\Phi_i
\]

(3.4-3)

where \( \Phi_i \) is a solution of the harmonic equation

\[
\nabla^2 \Phi_i = 0
\]

(3.4-4)

A particular solution of Eqs. (3.4-1) was suggested by Papkovich [5] and Goodier [6] in the form

\[
u_i^T = \frac{\partial \psi}{\partial x_i}
\]

(3.4-5)

where \( \psi \) is a scalar function satisfying the equation

\[
\nabla^2 \psi = \frac{1 + \nu}{1 - \nu} \alpha (T - T_0)
\]

(3.4-6)
Combining Eq. (3.4-3) with Eq. (3.4-5), the total solution of Eqs. (3.4-1) reduces to

\[ u_i = \frac{\partial}{\partial x_i} (\psi + x_k \Phi_k) - 4(1 - \nu) \Phi_i \]  
(3.4-7)

This method of solution satisfies all the conditions for both a simply-connected region as well as a multiply-connected region. The function \( \psi \) is called the *displacement potential*, and represents a particular solution in thermoelasticity problems.

In solving quasi-static thermoelasticity problems by the displacement potential method, we should obtain a solution for the displacement potential \( \psi \). This step of solution is performed after the temperature function has been determined from the heat transfer equations and the thermal boundary conditions. The displacements and stresses obtained in this step will generally not satisfy the boundary conditions. To complement the solution, the general solution for displacement corresponding to isothermal elasticity, \( U_i^g \), must be added to the displacement components obtained from the displacement potential. The constants of integration involved in the general solution are to be determined in such a manner that the total solution, the sum of \( U_i^g \) and \( U_i^T \), satisfies all the boundary conditions.

**Plane stress**

Let us now apply the results obtained above to the simple case of plane stress problems. Equations (3.4-1) in this case reduce to

\[
\begin{align*}
\frac{1 - \nu}{1 + \nu} \nabla^2 u + \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 2\alpha \frac{\partial T}{\partial x} \\
\frac{1 - \nu}{1 + \nu} \nabla^2 v + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 2\alpha \frac{\partial T}{\partial y}
\end{align*}
\]  
(3.4-8)

By introducing the displacement potential \( \psi \) through the relations

\[ u = \frac{\partial \psi}{\partial x}, \quad v = \frac{\partial \psi}{\partial y} \]  
(3.4-9)

Eqs. (3.4-8) become

\[
\begin{align*}
\frac{\partial}{\partial x} \left[ \frac{2}{1 + \nu} \nabla^2 \psi - 2\alpha(T - T_0) \right] &= 0 \\
\frac{\partial}{\partial y} \left[ \frac{2}{1 + \nu} \nabla^2 \psi - 2\alpha(T - T_0) \right] &= 0
\end{align*}
\]  
(3.4-10)

Equations (3.4-10) are both satisfied if

\[ \nabla^2 \psi = (1 + \nu)\alpha(T - T_0) \]  
(3.4-11)
4. General Solution of Thermoelastic Problems

The stress-displacement relations for the plane stress condition, see Eq. (1.12-2), reduce to

\[
\sigma_{xx} = \frac{E}{1 - \nu^2} \left[ \left( \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} \right) - (1 + \nu) \alpha (T - T_0) \right]
\]
\[
\sigma_{yy} = \frac{E}{1 - \nu^2} \left[ \left( \frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} \right) - (1 + \nu) \alpha (T - T_0) \right]
\]
\[
\sigma_{xy} = \frac{E}{2(1 + \nu)} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)
\]  \hspace{1cm} (3.4-12)

which, upon substitution of Eqs. (3.4-9), and using Eq. (3.4-11), become

\[
\sigma_{xx} = -2G \frac{\partial^2 \psi}{\partial y^2}
\]
\[
\sigma_{yy} = -2G \frac{\partial^2 \psi}{\partial x^2}
\]
\[
\sigma_{xy} = G \frac{\partial^2 \psi}{\partial x \partial y}
\]  \hspace{1cm} (3.4-13)

The stresses obtained through Eqs. (3.4-13) will clearly not satisfy the boundary conditions. To complete the formulation, the general solution corresponding to isothermal elasticity must be added to the stresses in Eqs. (3.4-13). To incorporate the general solution, we may use either Eqs. (3.4-3) and (3.4-4), or, instead, the complementary solution given by Airy stress function. In the latter case, the biharmonic equation representing the general solution of an isothermal elasticity problem in terms of Airy stress function is

\[
\nabla^4 \Phi = 0
\]  \hspace{1cm} (3.4-14)

where \( \Phi \) is Airy stress function. The relations between the stresses and Airy stress function in the plane stress condition are

\[
\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}
\]
\[
\sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}
\]
\[
\sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}
\]  \hspace{1cm} (3.4-15)

Therefore, the complete solution for a thermoelasticity problem in terms of Airy stress function \( \Phi \) and the displacement potential \( \psi \) for the plane stress condition becomes

\[
\sigma_{xx} = \frac{\partial^2}{\partial y^2} (\Phi - 2G \psi)
\]
\[ \sigma_{yy} = \frac{\partial^2}{\partial x^2} (\Phi - 2G\psi) \]
\[ \sigma_{xy} = -\frac{\partial^2}{\partial x \partial y} (\Phi - 2G\psi) \] (3.4-16)

**Plane strain**

For the plane strain condition, when \( \epsilon_{zz} = 0 \), the equilibrium equations in terms of displacement components reduce to

\[
(1 - 2\nu)\nabla^2 u + \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 2(1 + \nu)\alpha \frac{\partial T}{\partial x}
\]
\[
(1 - 2\nu)\nabla^2 v + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 2(1 + \nu)\alpha \frac{\partial T}{\partial y} \] (3.4-17)

which upon substitution of Eqs. (3.4-9) yields

\[
\nabla^2 \psi = \frac{1 + \nu}{1 - \nu} \alpha (T - T_0) \] (3.4-18)

To find the relations between the stresses and the displacement potential, we take the stress-strain relations for the plane strain conditions,

\[
\sigma_{xx} = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu) \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} - (1 + \nu)\alpha (T - T_0) \right]
\]
\[
\sigma_{yy} = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu) \frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} - (1 + \nu)\alpha (T - T_0) \right]
\]
\[
\sigma_{xy} = G \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \] (3.4-19)

Substituting Eqs. (3.4-9) and Eq. (3.4-18) into Eq. (3.4-19), we get

\[
\sigma_{xx} = -2G \frac{\partial^2 \psi}{\partial y^2}
\]
\[
\sigma_{yy} = -2G \frac{\partial^2 \psi}{\partial x^2}
\]
\[
\sigma_{xy} = 2G \frac{\partial^2 \psi}{\partial x \partial y} \] (3.4-20)

The complete solution in terms of the displacement potential \( \psi \) and the stress function \( \Phi \) is obtained from Eqs. (3.4-16).

The axial stress in the plane strain condition, when \( \epsilon_{zz} = 0 \), is obtained by noting that

\[ \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) - E\alpha (T - T_0) \] (3.4-21)

which in terms of the displacement potential \( \psi \) is

\[ \sigma_{zz} = -2G\nabla^2 \psi \] (3.4-22)
Equations (3.4-16) in polar coordinates are

\[\sigma_{rr} = \left(\frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}\right)(\Phi - 2G\psi)\]
\[\sigma_{\phi\phi} = \frac{\partial^2}{\partial r^2}(\Phi - 2G\psi)\]
\[\sigma_{r\phi} = -\frac{1}{r} \frac{\partial}{\partial \phi}\left[\frac{1}{r} \frac{\partial}{\partial \phi}(\Phi - 2G\psi)\right]\]  

(3.4-23)

These formulas may be used for both plane stress and plane strain problems.

5 Solution of Two-Dimensional Navier Equations

Navier equations in the two-dimensional thermoelasticty for the steady-state temperature distribution may be solved by a series expansion. The governing steady-state thermoelastic equations for the plane strain condition are

\[2(1 - \nu)\frac{\partial^2 u}{\partial x^2} + (1 - 2\nu)\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} = 2\alpha(1 + \nu)\frac{\partial \theta}{\partial x}\]
\[2(1 - \nu)\frac{\partial^2 v}{\partial y^2} + (1 - 2\nu)\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} = 2\alpha(1 + \nu)\frac{\partial \theta}{\partial y}\]
\[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0\]  

(3.5-1)

where \(\theta = T - T_0\) is the temperature change. Let us assume that the set of Eqs. (3.5-1) model a problem which exhibits some periodic nature in \(x\) direction. In this case the dependent functions \(u, v,\) and \(\theta\) are expanded into a Fourier sine and cosine series as follows from [7]. Introducing the Fourier sine and cosine transformation, respectively, by

\[\hat{f}(x,y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(x,y) \sin mxdx\]
\[\hat{f}(x,y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(x,y) \cos mxdx\]  

(3.5-2)

Eqs. (3.5-1) are transformed to the following form

\[(D^2 - K^2m^2)\ddot{u} - (K^2 - 1)mD\dot{v} = -\beta m\dot{\theta}\]
\[(K^2D^2 - m^2)\ddot{v} + (K^2 - 1)mD\ddot{u} = \beta m\dot{\theta}\]
\[(D^2 - m^2)\ddot{\theta} = 0\]  

(3.5-3)
where
\[ D = \frac{d}{dy} \quad \beta = \frac{2(1 + \nu)}{1 - 2\nu} \alpha \quad K^2 = \frac{2(1 - \nu)}{1 - 2\nu} \] (3.5-4)

Equations (3.5-3) represent a system of ordinary differential equations which may be solved for the transformed functions \( \bar{u}, \hat{v}, \) and \( \hat{\theta} \). The solution happens to be
\[
\bar{u} = (A_1 + myA_2) \cosh my + (B_1 + myB_2) \sinh my \\
\hat{v} = (A_3 + myA_4) \cosh my + (B_3 + myB_4) \sinh my \\
\hat{\theta} = A \cosh my + B \sinh my
\] (3.5-5)

where the coefficients \( A, B, A_1 \) to \( A_4 \), and \( B_1 \) to \( B_4 \) are the constants of integration and are functions of \( m \) and depend on the boundary conditions. These constants, however, are not independent of each other. Substituting the solution (3.5-5) into the first of Eqs. (3.5-3) and equating the coefficients of \( \cosh my, \sinh my, my \cosh my, \) and \( my \sinh my \) yields
\[
A_2 = -B_4 = \frac{1}{m(k^2 + 1)}[m(K^2 - 1)(A_3 + B_1) - \beta B] \\
A_4 = -B_2 = \frac{-1}{m(k^2 + 1)}[m(K^2 - 1)(A_1 + B_3) - \beta A]
\] (3.5-6)

The constants \( A_2, A_4, B_2, \) and \( B_4 \) are eliminated from Eqs. (3.5-5) by substitution from Eqs. (3.5-6).

Recalling the stress-strain relations for the plane strain condition
\[
\sigma_{xx} = \frac{2G}{1 - 2\nu}[(1 - \nu)\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} - (1 - \nu)\alpha \theta] \\
\sigma_{yy} = \frac{2G}{1 - 2\nu}[(1 - \nu)\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x} - (1 - \nu)\alpha \theta] \\
\sigma_{xy} = G(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})
\] (3.5-7)

we receive the general solution for stresses in terms of the constants of integration
\[
\hat{\sigma}_{xx}(x, y) = G\{[K^2(A_1 + myA_2 + A_4)m - 2A_4m \\
+(k^2 - 2)(B_3 + myB_4)m - \beta A] \cosh my + [K^2m(B_1 + myB_2 + B_4) \\
- 2B_4m + m(K^2 - 2)(A_3 + myA_4) - \beta B] \sinh my\}
\]
\[
\hat{\sigma}_{yy}(x, y) = G\{[m(K^2 - 2)(A_1 + myA_2) + K^2m(A_4 \\
+ B_3 + myB_4) - \beta A] \cosh my + [m(K^2 - 2)(B_1 + myB_2) \\
+ mK^2(B_4 + A_3 + myA_4) - \beta B] \sinh my\}
\]
\[
\sigma_{xy}(x, y) = Gm[(A_1 + 2myA_2 + B_2 - B_3) \sinh my \\
+ (B_1 + 2myB_2 + A_2 - A_3) \cosh my]
\] (3.5-8)

The constants of integration are now obtained using the given kinematical and forced boundary conditions, as well as the thermal boundary conditions. The given kinematical boundary conditions must satisfy the first two of Eqs. (3.5-5). The thermal boundary conditions should be applied to the third of Eqs. (3.5-5). The stress boundary conditions should be applied to Eqs. (3.5-8).

Once the constants of integration are obtained, the inverse Fourier transformation is applied and the functions are transformed to the original domain.

As an example, a plane strain problem of an infinite, homogeneous, isotropic elastic layer occupying the region \(0 \leq y \leq h\) is considered. The mechanical boundary conditions may be considered as in [7]

\[
\begin{align*}
\sigma_{yy} &= \sigma_{xy} = 0 \quad \text{on} \quad y = 0 \\
v &= \sigma_{xy} = 0 \quad \text{on} \quad y = h
\end{align*}
\] (3.5-9)

The thermal boundary conditions are

\[
\begin{align*}
\theta &= f(x) \quad a \leq |x| \leq b \quad y = 0 \\
\frac{\partial \theta}{\partial y} &= 0 \quad 0 \leq |x| \leq a \quad |x| > b \quad y = 0 \\
\frac{\partial \theta}{\partial y} &= 0 \quad 0 \leq |x| < \infty \quad y = h
\end{align*}
\] (3.5-10)

For the given mechanical and thermal boundary conditions, the constants of integration are obtained to be

\[
\begin{align*}
A_2 &= 0 \quad A_4 = 0 \quad B_2 = 0 \quad B_4 = 0 \\
A_3 &= B_1 = -\frac{1}{2} \frac{\beta m^{-1} A(m) \tanh mh}{K^2 - 1} \\
A_1 &= B_3 = \frac{1}{2} \frac{\beta m^{-1} A(m)}{K^2 - 1}
\end{align*}
\] (3.5-11)

The constant \(A(m)\) is to be found from the second condition of Eqs. (3.5-10). Substituting the constants of integration from Eq. (3.5-11) into Eq. (3.5-5) yields

\[
\begin{align*}
\bar{u} &= \frac{\beta A(m) \cosh m(y - h)}{2m(K^2 - 1) \cosh mh} \\
\hat{v} &= \frac{\beta A(m) \sinh m(y - h)}{2m(K^2 - 1) \cosh mh} \\
\hat{\theta} &= \frac{A(m) \cosh m(y - h)}{\cosh mh}
\end{align*}
\] (3.5-12)
The inverse transformation of Eqs. (3.5-12) provides the expressions for physical quantities of the displacements and the temperature in the original domain. They are

\[ u(x, y) = \frac{\beta}{\sqrt{2\pi(K^2 - 1)}} \int_0^\infty A(m) \frac{\cosh m(h - y)}{\cosh mh} \sin mxdm \]

\[ v(x, y) = \frac{\beta}{\sqrt{2\pi(K^2 - 1)}} \int_0^\infty A(m) \frac{\sinh m(h - y)}{\cosh mh} \cos mxdm \]

\[ \theta(x, y) = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \int_0^\infty A(m) \frac{\cosh m(h - y)}{\cosh mh} \cos mxdm \] \hspace{1cm} (3.5-13)

### 6 General Solution in Cylindrical Coordinates

Consider the asymmetric three-dimensional thermoelastic problems in cylindrical coordinates \((r, \phi, z)\). The equation of motion (1.3-8) for the static condition in three-dimensional cylindrical coordinates in the absence of body forces are [8–12]

\[ \sigma_{rr,r} + r^{-1}\sigma_{\phi r,\phi} + \sigma_{zr,z} + r^{-1}(\sigma_{rr} - \sigma_{\phi\phi}) = 0 \]

\[ \sigma_{r\phi,r} + r^{-1}\sigma_{\phi\phi,\phi} + \sigma_{z\phi,z} + 2r^{-1}\sigma_{r\phi} = 0 \]

\[ \sigma_{rz,r} + r^{-1}\sigma_{\phi z,\phi} + \sigma_{zz,z} + r^{-1}\sigma_{rz} = 0 \] \hspace{1cm} (3.6-1)

The stress-strain relations are

\[ \sigma_{rr} = (2\mu + \lambda)\epsilon_{rr} + \lambda(\epsilon_{\phi\phi} + \epsilon_{zz}) - \beta(T - T_0) \]

\[ \sigma_{\phi\phi} = (2\mu + \lambda)\epsilon_{\phi\phi} + \lambda(\epsilon_{rr} + \epsilon_{zz}) - \beta(T - T_0) \]

\[ \sigma_{zz} = (2\mu + \lambda)\epsilon_{zz} + \lambda(\epsilon_{rr} + \epsilon_{\phi\phi}) - \beta(T - T_0) \]

\[ \sigma_{rz} = 2\mu\epsilon_{rz} \]

\[ \sigma_{\phi z} = 2\mu\epsilon_{\phi z} \]

\[ \sigma_{r\phi} = 2\mu\epsilon_{r\phi} \] \hspace{1cm} (3.6-2)

where \( \beta = (3\lambda + 2\mu)\alpha \). The strain-displacement relations in cylindrical coordinates are

\[ \epsilon_{rr} = u,r \]

\[ \epsilon_{\phi\phi} = r^{-1}(u + v,\phi) \]

\[ \epsilon_{zz} = w,z \]

\[ \epsilon_{r\phi} = \frac{1}{2}(r^{-1}u,\phi + v,r - r^{-1}v) \]

\[ \epsilon_{\phi z} = \frac{1}{2}(v,z + r^{-1}w,\phi) \]

\[ \epsilon_{rz} = \frac{1}{2}(u,z + w,r) \] \hspace{1cm} (3.6-3)
6. General Solution in Cylindrical Coordinates

Substituting Eqs. (3.6-3) into (3.6-2) and finally into (3.6-1) results in the equilibrium equations in terms of displacement components

\[
(2\mu + \lambda)(u_{,rr} + r^{-1}u_{,r} - r^{-2}u) + \mu r^{-2}u_{,\phi\phi} + \mu u_{,zz} + (\lambda + \mu)r^{-1}u_{,r}\phi
- \frac{3\mu + \lambda}{r^2}v_{,\phi} + (\lambda + \mu)w_{,rz} = \beta T_r
\]

\[
\mu(v_{,rr} + r^{-1}v_{,r} - r^{-2}v) + (2\mu + \lambda)r^{-2}v_{,\phi\phi} + \mu v_{,zz} + (\lambda + \mu)r^{-1}v_{,z\phi} = \beta r^{-1}T_\phi
\]

\[
u(w_{,rr} + r^{-1}w_{,r} - r^{-2}w_{,\phi\phi}) + (2\mu + \lambda)w_{,zz} + (\lambda + \mu)(r^{-1}v_{,\phi z}
+ 2r^{-1}u_{,z}) = \beta T_z
\]

(3.6-4)

Based on the displacement formulations, the solution of the governing equations reduces to the following scalar equations for the thermoelastic displacement potential \( \psi \), Michell’s function \( M \), and the so called Boussinesq’s function \( B \),

\[
\nabla^2 \psi = \frac{1 + \nu}{1 - \nu} \alpha(T - T_0)
\]

\[
\nabla^4 M = 0
\]

\[
\nabla^2 B = 0
\]

(3.6-5) (3.6-6) (3.6-7)

The displacement components are related to these functions. They are

\[
u = \psi_{,r} - M_{,rz} + \frac{2}{r} B_{,\phi}
\]

\[
v = \frac{1}{r} \psi_{,\phi} - \frac{1}{r} M_{,z\phi} - 2B_{,r}
\]

\[
w = \psi_{,z} + 2(1 - \nu)\nabla^2 M - M_{,zz}
\]

(3.6-8)

where

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}
\]

(3.6-9)

Substituting the displacement components of Eqs. (3.6-8) into the strain-displacement relations (3.6-3) and the stress-strain relations (3.6-2), results in the following equations for stresses

\[
\sigma_{rr} = 2G[\psi_{,rr} - \nabla^2 \psi + (\nu \nabla^2 M - M_{,rr})_{,z} + \frac{2}{r} B_{,r\phi} - \frac{2}{r^2} B_{,\phi}]
\]

\[
\sigma_{\phi\phi} = 2G[\frac{1}{r} \psi_{,r} + \frac{1}{r^2} \psi_{,\phi\phi} - \nabla^2 \psi + (\nu \nabla^2 M - M_{,r})_{,\phi}
- \frac{1}{r^2} M_{,\phi\phi})_{,z} - \frac{2}{r} B_{,r\phi} + \frac{2}{r^2} B_{,\phi}]
\]
\[
\sigma_{zz} = 2G\{\psi_{zz} - \nabla^2\psi + [(2 - \nu)\nabla^2M - M_{zz},z]\}
\]
\[
\sigma_{r\phi} = 2G\left[\frac{1}{r}\psi_{r\phi} - \frac{1}{r^2}\psi_{,\phi} + \frac{1}{r}\left(\frac{M}{r}\right)
- M_{,r},\phi z - B_{,rr} + \frac{1}{r}B_{,r} + \frac{1}{r^2}B_{,\phi\phi}\right]
\]
\[
\sigma_{rz} = 2G\{\psi_{rz} + [(1 - \nu)\nabla^2M - M_{zz},r] + \frac{1}{r}B_{,\phi z}\}
\]
\[
\sigma_{z\phi} = 2G\left\{\frac{1}{r}\psi_{z\phi} + \frac{1}{r}[(1 - \nu)\nabla^2M - M_{zz},\phi - B_{,rz}]\right\}
\] (3.6-10)

where \(G = \mu\) is the shear modulus.

Once the displacement potential \(\psi\), Michell’s function \(M\), and Boussinesq’s functions \(B\) are found, the displacements and the stresses are obtained from Eqs. (3.6-8) and (3.6-10), respectively.

The general solution of the harmonic and biharmonic equations (3.6-5) to (3.6-7) may be obtained using the method of separation of variables. A very general solution of a harmonic equation may be written as shown in [11]. The solution of a harmonic equation in general cylindrical coordinates in terms of the variables \(r\), \(\phi\), and \(z\) is a combination of each group of solutions, as given below. Selection of each group of the solution depends upon the boundary conditions.

\[
\begin{align*}
\left\{ J_n(\beta r) \right\} & \left\{ \cos n\phi \right\} \left\{ \cosh \beta z \right\} \left\{ J_0(\beta r) \right\} \left\{ \phi \right\} \left\{ \cosh \beta z \right\} \\
\left\{ Y_n(\beta r) \right\} & \left\{ \sin n\phi \right\} \left\{ \sinh \beta z \right\} \left\{ Y_0(\beta r) \right\} \left\{ 1 \right\} \left\{ \sinh \beta z \right\}
\end{align*}
\]

\[
\begin{align*}
\left\{ I_n(\beta r) \right\} & \left\{ \cos n\phi \right\} \left\{ \cos \beta z \right\} \left\{ I_0(\beta r) \right\} \left\{ \phi \right\} \left\{ \cos \beta z \right\} \\
\left\{ K_n(\beta r) \right\} & \left\{ \sin n\phi \right\} \left\{ \sin \beta z \right\} \left\{ K_0(\beta r) \right\} \left\{ 1 \right\} \left\{ \sin \beta z \right\}
\end{align*}
\]

\[
\begin{align*}
\left\{ r^n \right\} & \left\{ \cos n\phi \right\} \left\{ z \right\} \left\{ \ln r \right\} \left\{ \phi \right\} \left\{ z \right\} \\
\left\{ r^{-n} \right\} & \left\{ \sin n\phi \right\} \left\{ 1 \right\} \left\{ 1 \right\} \left\{ 1 \right\} \left\{ 1 \right\}
\end{align*}
\] (3.6-11)

As an example, a solution of harmonic partial differential equation in cylindrical coordinates with homogeneous boundary conditions along the \(z\)-axis is

\[
T(r, \phi, z) = \sum_n [A_n I_n(\beta_n r) + B_n K_n(\beta_n r)][C_n \cos n\phi + D_n \sin n\phi]
\]

\[
\times [E_n \cos \beta_n z + F_n \sin \beta_n z]
\]

If the boundary conditions along the \(z\)-axis are nonhomogeneous, the third bracket may be replaced with

\[
[E_n \cosh \beta_n z + F_n \sinh \beta_n z]
\]

Similarly, a very general solution of a biharmonic function may be written as the above plus the functions
7. Solution of Problems in Spherical Coordinates

\[ \begin{aligned}
\{ \beta r J_{n+1}(\beta r) \} &= \begin{cases} \cos n\phi \\ \sin n\phi \end{cases} \begin{cases} \cosh \beta z \\ \sinh \beta z \end{cases} \begin{cases} J_n(\beta r) \\ Y_n(\beta r) \end{cases} \begin{cases} \cos n\phi \\ \sin n\phi \end{cases} \begin{cases} \beta z \sinh \beta z \\ \beta z \cosh \beta z \end{cases} \\
\{ \beta r I_1(\beta r) \} &= \begin{cases} \phi \\ 1 \end{cases} \begin{cases} \cos \beta z \\ \sin \beta z \end{cases} \begin{cases} I_0(\beta r) \\ K_0(\beta r) \end{cases} \begin{cases} \phi \\ 1 \end{cases} \begin{cases} \beta z \sin \beta z \\ \beta z \cos \beta z \end{cases} \\
\{ \beta r K_{n+1}(\beta r) \} &= \begin{cases} \cos n\phi \\ \sin n\phi \end{cases} \begin{cases} \cos \beta z \\ \sin \beta z \end{cases} \begin{cases} I_n(\beta r) \\ K_n(\beta r) \end{cases} \begin{cases} \cos n\phi \\ \sin n\phi \end{cases} \begin{cases} \beta z \sin \beta z \\ \beta z \cos \beta z \end{cases} \\
\{ \beta r I_1(\beta r) \} &= \begin{cases} \phi \\ 1 \end{cases} \begin{cases} \cos \beta z \\ \sin \beta z \end{cases} \begin{cases} I_0(\beta r) \\ K_0(\beta r) \end{cases} \begin{cases} \phi \\ 1 \end{cases} \begin{cases} \beta z \sin \beta z \\ \beta z \cos \beta z \end{cases} \\
\left( r^{n+2} \begin{cases} \cos n\phi \\ \sin n\phi \end{cases} \begin{cases} z \\ 1 \end{cases} \begin{cases} r^n \\ r^{-n} \end{cases} \begin{cases} \cos n\phi \\ \sin n\phi \end{cases} \begin{cases} z^3 \\ r^3 \end{cases} \begin{cases} r \ln r \\ \sin \phi \end{cases} \begin{cases} 1 \end{cases} \end{align} \]

\[ \begin{aligned}
\left( r^2 \begin{cases} \cos n\phi \\ \sin n\phi \end{cases} \begin{cases} z \\ 1 \end{cases} \begin{cases} \ln r \\ 1 \end{cases} \begin{cases} \phi \\ 1 \end{cases} \begin{cases} z^3 \\ z^2 \end{cases} \begin{cases} 0 \phi \cos \phi \end{cases} \begin{cases} \sin \phi \cos \phi \end{cases} \begin{cases} 1 \end{cases} \end{align} \]

(3.6-12)

Solution of any harmonic and biharmonic equation can be obtained by proper combination of the functions listed in Eqs. (3.6-11) and (3.6-12).

7 Solution of Problems in Spherical Coordinates

Consider the equilibrium equations in spherical coordinate system \((r, \theta, \phi)\), as shown in Fig. 3.7-1. The equations in dimensionless form and in terms of the displacement components are \([13,14]\)

\[ \frac{2(1 - \nu)}{1 - 2\nu} \bar{e}_{\rho} + \Delta \bar{u}_r - \frac{1}{\rho} (\rho^2 \bar{u}_{r,\rho})_{\rho} - \frac{1}{\rho^2 \sin \theta} [\rho \sin \theta \bar{u}_\theta]_{\theta} \]

\[ + (\rho \bar{u}_\phi)_{\phi,\rho} - \frac{2(1 + \nu)}{1 - 2\nu} \bar{T}_{\rho} = 0 \]

\[ \frac{1}{\rho} \left[ \frac{2(1 - \nu)}{1 - 2\nu} \bar{e} - \bar{u}_{r,\rho} - \frac{2(1 + \nu)}{1 - 2\nu} \bar{T}_{\rho} \right]_{\theta} + \frac{1}{\rho^2} (\rho^2 \bar{u}_{\theta,\rho})_{\rho} \]

\[ + \frac{1}{\rho^2 \sin^2 \theta} [\bar{u}_{\theta,\phi} - (\sin \theta \bar{u}_\phi)_{\theta}]_{\phi} = 0 \]

\[ \frac{1}{\rho \sin \theta} \left[ \frac{2(1 - \nu)}{1 - 2\nu} \bar{e} - \bar{u}_{r,\rho} - \frac{2(1 + \nu)}{1 - 2\nu} \bar{T}_{\rho} \right]_{\phi} + \frac{1}{\rho^2} (\rho^2 \bar{u}_{\phi,\rho})_{\rho} \]

\[ - \frac{1}{\rho^2} \left\{ \frac{1}{\sin \theta} [\bar{u}_{\theta,\phi} - (\sin \theta \bar{u}_\phi)_{\theta}] \right\}_{\theta} = 0 \]

(3.7-1) (3.7-2) (3.7-3)
where the dimensionless quantities are defined as
\[
\rho = \frac{r}{a} \quad \tilde{e} = \frac{e}{\alpha T_0} \quad \tilde{T} = \frac{T - T_0}{T_0} \quad \tilde{u}_i = \frac{u_i}{a\alpha T_0} \quad i = r, \theta, \phi \quad (3.7-4)
\]
where \(a\) is some characteristic length and \(T_0\) is the reference temperature. The dimensionless dilatation and Laplace operator \(\tilde{\Delta}\) are defined as
\[
\tilde{e} = \frac{1}{\rho^2} (\rho^2 \tilde{u}_r, \rho) + \frac{1}{\rho \sin \theta} [(\sin \theta \tilde{u}_\theta, \theta) + \tilde{u}_\phi, \phi] \quad (3.7-5)
\]
\[
\tilde{\Delta} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 \frac{\partial}{\partial \rho}) + \frac{1}{\rho^2 \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right] \quad (3.7-6)
\]

Now, \(\partial/\partial \phi\) is applied to Eq. (3.7-2), then Eq. (3.7-3) is multiplied by \(\sin \theta\) and \(\partial/\partial \theta\) is applied to it. Adding the two resulting equations, \(\tilde{e}, \tilde{u}_r,\) and \(\tilde{T}\) are eliminated and the result is
\[
\tilde{\Delta}\left\{ \frac{1}{\sin \theta} [\tilde{u}_\theta, \phi - (\sin \theta \tilde{u}_\phi, \theta)] \right\} = 0 \quad (3.7-7)
\]
Introducing the thermoelastic displacement potential \(\psi\) by
\[
\tilde{u}_\theta = \frac{1}{\rho} \psi, \theta \quad \tilde{u}_\phi = \frac{1}{\rho \sin \theta} \psi, \phi \quad (3.7-8)
\]
Eq. (3.7-7) is automatically satisfied by the definition of the thermoelastic displacement potential \(\psi\) given by Eq. (3.7-8). To obtain the relation between \(\tilde{u}_r\) and \(\psi\), Eq. (3.7-8) is substituted into Eq. (3.7-1)
\[
\frac{2(1 - \nu)}{1 - 2\nu} (\tilde{e} - \frac{1 + \nu}{1 - \nu} \tilde{T}), \rho + \frac{1}{\rho^2 \sin \theta} \left\{ [\sin \theta (\tilde{u}_r - \psi, \rho), \theta]_\theta + \frac{1}{\sin \theta} (\tilde{u}_r - \psi, \rho), \phi \phi \right\} = 0 \quad (3.7-9)
\]
Dividing Eq. (3.7-9) by \( \rho \), gives

\[
\frac{2(1 - \nu)}{1 - 2\nu} \frac{1}{\rho} \left( \ddot{e} - \frac{1 + \nu}{1 - \nu} \ddot{T} \right)_{,\rho} + \frac{1}{\rho^2 \sin \theta} \left\{ \sin \theta \left( \frac{1}{\rho} \bar{u}_r - \frac{1}{\rho} \psi_{,\rho} \right)_{,\theta} \right\} = 0
\]

(3.7-10)

The first term of Eq. (3.7-10) is related to Boussinesq’s function \( B \) by

\[
\ddot{e} - \frac{1 + \nu}{1 - \nu} \ddot{T} = \frac{1 + \nu}{1 - \nu} (\rho B)_{,\rho}
\]

(3.7-11)

It is easily checked that the following relation is identically satisfied for any function \( F \)

\[
\frac{1}{\rho} (\rho F)_{,\rho\rho} = \frac{1}{\rho^2} (\rho^2 F_{,\rho})_{,\rho}
\]

(3.7-12)

On this basis, and from the definition of Boussinesq’s function \( B \), Eq. (3.7-10) is rewritten as

\[
\frac{2(1 + \nu)}{1 - 2\nu} \frac{1}{\rho^2} (\rho^2 B_{,\rho})_{,\rho} + \frac{1}{\rho^2 \sin \theta} \left\{ \sin \theta \left( \frac{1}{\rho} \bar{u}_r - \frac{1}{\rho} \psi_{,\rho} \right)_{,\theta} \right\} = 0
\]

(3.7-13)

Assuming the following relation for the functions \( B, \bar{u}_r, \) and \( \psi \)

\[
\frac{2(1 + \nu)}{1 - 2\nu} B = \frac{1}{\rho} (\bar{u}_r - \psi_{,\rho})
\]

(3.7-14)

then Eq. (3.7-13) gives

\[
\ddot{\Delta} B = 0
\]

(3.7-15)

From Eq. (3.7-14) the expression for \( \bar{u}_r \) is

\[
\bar{u}_r = \psi_{,\rho} + \frac{2(1 + \nu)}{1 - 2\nu} \rho B
\]

(3.7-16)

Substituting Eqs. (3.7-8) and (3.7-16) into Eq. (3.7-5), the expression for dilatation becomes

\[
\ddot{e} = \ddot{\Delta} \psi + \frac{2(1 + \nu)}{1 - 2\nu} \frac{1}{\rho^2} (\rho^3 B)_{,\rho}
\]

(3.7-17)

Substituting for \( \ddot{e} \) in Eq. (3.7-11), the condition which the thermoelastic displacement potential \( \psi \) has to satisfy is obtained to be

\[
\ddot{\Delta} \psi = \frac{1 + \nu}{1 - \nu} [\dddot{T} - \frac{1}{1 - 2\nu} (\rho B)_{,\rho} - \frac{4(1 - \nu)}{1 - 2\nu} B]
\]

(3.7-18)
The displacement components are all related to the functions $\psi$ and $B$. Once the solution of Eqs. (3.7-15) and (3.7-18) are obtained for $B$ and $\psi$, the displacement components are calculated. It may be verified that the displacement components in terms of the functions $\psi$ and $B$ automatically satisfy the equilibrium equations (3.7-1) to (3.7-3).

From stress-strain and strain-displacement relations it follows that the stress components are related to the functions $\psi$ and $B$ and they are

$$
\bar{\sigma}_{rr} = \frac{1}{1+\nu} \psi_{,pp} + \frac{2 - \nu}{(1 - \nu)(1 - 2\nu)} (\rho B)_{,\rho} - \frac{1}{1 - \nu} \bar{T} \\
\bar{\sigma}_{\theta\theta} = \frac{1}{1+\nu} \left[ -\frac{1}{\rho} \psi_{,\rho} + \frac{1 - \mu^2}{\rho^2} \psi_{,\mu} - \frac{\mu}{\rho^2} \psi_{,\mu} \right] + \frac{\nu}{(1 - \nu)(1 - 2\nu)} \rho B_{,\rho} + \frac{2 - \nu}{(1 - \nu)(1 - 2\nu)} B - \frac{1}{1 - \nu} \bar{T} \\
\bar{\sigma}_{\phi\phi} = \frac{1}{1+\nu} \left[ -\frac{1}{\rho} \psi_{,\rho} - \frac{\mu}{\rho^2} \psi_{,\mu} + \frac{1}{\rho^2(1 - \mu^2)} \psi_{,\phi} \right] + \frac{\nu}{(1 - \nu)(1 - 2\nu)} \rho B_{,\rho} + \frac{2 - \nu}{(1 - \nu)(1 - 2\nu)} B - \frac{1}{1 - \nu} \bar{T} \\
\bar{\sigma}_{r\theta} = -(1 - \mu^2)^{\frac{1}{2}} \left[ \frac{1}{1 - \nu} \left( \frac{\psi}{\rho} \right)_{,\rho} + \frac{1}{1 - 2\nu} B_{,\mu} \right] \\
\bar{\sigma}_{\theta\phi} = -\frac{1}{1+\nu} \frac{1}{\rho^2} \left[ \psi_{,\mu} + \frac{\mu}{1 - \mu^2} \psi \right] \\
\bar{\sigma}_{\phi r} = \frac{1}{(1 - \mu^2)^{\frac{1}{2}}} \left[ \frac{1}{1+\nu} \left( \frac{\psi}{\rho} \right)_{,\rho} + \frac{1}{1 - 2\nu} B \right]
$$

(3.7-19)

where the dimensionless stress is $\bar{\sigma}_{ij} = \sigma_{ij}/E\alpha T_0$ and $\mu = \cos \theta$.

8 Problems

1. Prove that in $xyz$-system $u_i' = \frac{\partial}{\partial x_i} (x_k \Phi_k) - 4(1 - \nu)\Phi_i$ is the general solution and $u_i' = \psi_{,i}$ is the particular solution of the three-dimensional Navier equations.

2. Prove Eqs. (3.6-4) using the stress-strain and strain-displacement relations in cylindrical coordinates.

3. Derive equivalent relations to Eqs. (3.6-4) for the plane stress and plane strain conditions.

4. Derive relations for the stresses, the stress function, and the displacement potential in cylindrical coordinates for the plane stress and the plane strain conditions.
5. Verify Eqs. (3.6-5) to (3.6-7).

6. Verify the solution of Eqs. (3.5-3) given by Eqs. (3.5-5).

7. Derive the equilibrium equations in terms of the displacement components in spherical coordinates, Eqs. (3.7-1) to (3.7-3).

8. Verify Eqs. (3.7-19).

Bibliography


